SIMPLICIAL HOMOLOGY OF STRONG SEMILATTICES OF BANACH ALGEBRAS

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ABSTRACT. Certain semigroups are known to admit a 'strong semi-lattice decomposition' into simpler pieces. We introduce a class of Banach algebras that generalise the ℓ^1-convolution algebras of such semi-groups, and obtain a disintegration theorem for their simplicial homology. Using this we show that for any Clifford semigroup S of amenable groups, $\ell^1(S)$ is simplicially trivial: this generalises results in [3]. Some other applications are presented.

1. INTRODUCTION

The connections between structural properties of a semigroup and cohomological properties of its ℓ^1-convolution algebra are still not fully understood. However, much work has been done (and much is known) for the case of group algebras. In attempting to extend existing techniques and results to more general semigroups, a promising class to consider first is that of so-called *Clifford semigroups*. Partial results have been obtained in low dimensions [1, 5] but hitherto there seems to have been no systematic attack on the higher-degree cohomology problems.

Clifford semigroups are known to arise as *strong semilattices of groups* (terminology which will be explained below). Recently it was established in [3] that the ℓ^1-convolution algebras of semilattices are simplicially trivial; in this article we generalise the techniques of [3] to deal with Clifford semi-groups. More precisely, for semigroups S which admit a 'strong semilattice decomposition', we obtain a 'disintegration' theorem for the simplicial homology of $\ell^1(S)$ (see Theorems 3.6, 3.8 below). As a corollary we prove that $\ell^1(S)$ is simplicially trivial whenever S is a Clifford semigroup with amenable constituent groups. (If S is a commutative Clifford semigroup, then as in [3] we may go on to deduce that cohomology of $\ell^1(S)$ vanishes in degrees 1 and above for any symmetric coefficient bimodule.)

Our disintegration result also covers other instances of semigroups with a strong semilattice decomposition: in particular we show that the ℓ^1-convolution algebras of normal bands are simplicially trivial.

So far we have not mentioned the 'strong semilattices of Banach algebras' of the title. These are most succinctly described, in the language of category

theory, as semilattice-shaped diagrams in the category of Banach algebras and contractive homomorphisms. We shall give a more concrete description below; for now it suffices to note that this construction

(i) gives rise to Banach algebras that include ℓ^1-convolution algebras of Clifford semigroups as special cases, and

(ii) seems to provide the correct level of generality for our disintegration theorem 3.8.

Our definition is slightly more general than the original one given in [2, §5.1].

We remark that in a recent preprint [4], Ghandehari, Hatami and Spronk have independently introduced a more general notion of a Banach algebra *graded over a semilattice*: however, they restrict attention to the case where the semilattice is finite (since they are primarily interested in amenability questions), while the present paper is motivated almost exclusively by the case where the indexing semilattice is infinite.

Before starting on the definitions, lemmas and theorems some general remarks on this article are in order. We will need to draw on background material from functional analysis: in particular we assume the reader is familiar with the projective tensor product of Banach spaces and elementary Banach space properties of ℓ^1. For homology and cohomology of Banach algebras we refer the reader to the two classic sources [6, 8].

We also assume familiarity with some basic category theory, such as may be found in [9]; this is not vital for the proofs to follow but provides some context for the underlying strategy.

2. Definitions and preliminaries

Since our eventual interest is in ℓ^1-convolution algebras of semigroups, we start by recalling some basic notions from semigroup theory.

Definition 2.1. A *semilattice* is a commutative semigroup where each element is idempotent.

Each semilattice has a canonical partial order on it, defined by

$$f \preceq e \iff ef = f$$

and so may be regarded as a poset in a natural way.

The following example will play an important role later; we introduce it here for emphasis. (Note that it differs slightly from the corresponding example in [3].)

Example 2.2 (Free semilattices)**.** Let $\mathbb{I}$ be a finite set and consider $2^{\mathbb{I}} \setminus \{\emptyset\}$, the collection of all non-empty subsets of $\mathbb{I}$. We may equip $2^{\mathbb{I}} \setminus \{\emptyset\}$ with the structure of a semilattice, by taking the product of two members A and B to be $A \cup B$; note that the canonical partial order is then given by

$$C \preceq D \iff C \cup D = C \iff C \supseteq D$$

We refer to the semilattice thus obtained as the *free semilattice generated by* $\mathbb{I}$. (One could extend this definition to the case where $\mathbb{I}$ is infinite, but for our purposes the finite case is enough.)

Semilattices have been used in the structure theory of semigroups; one fundamental instance of this is the concept of a strong semilattice of semigroups, which we now define.

Let $\mathcal{S}$ be a non-empty class of semigroups which is closed under taking semigroup homomorphisms. Alternatively, $\mathcal{S}$ may be viewed as a full subcategory of the category SGp of semigroups and semigroup homomorphisms.

Definition 2.3 (cf. [7, Ch. IV]). A *semilattice diagram in* $\mathcal{S}$ consists of: a semilattice L; a family $(S_e)_{e \in L}$ of semigroups in $\mathcal{S}$; and semigroup homomorphisms $\varphi_{f,e} : S_e \to S_f$ for each pair $(e, f) \in L \times L$ such that $e \succeq f$, which satisfy the following compatibility conditions:

- $\varphi_{e,e}$ is the identity homomorphism on S_e;
- if $e \succeq f \succeq g$ in L then $\varphi_{g,f} \circ \varphi_{f,e} = \varphi_{g,e}$.

To each such semilattice diagram we may associate a semigroup as follows: the underlying set is defined to be the disjoint union $\coprod_{e \in L} S_e$; and multiplication is defined by

$$x \cdot y := \varphi_{ef,e}(x)\varphi_{ef,f}(y) \qquad (x \in S_e, y \in S_f)$$

or, more pictorially, by

$$x \in \quad S_e \qquad\qquad\qquad S_f \quad \ni y$$
$$\varphi_{ef,e} \searrow \quad \swarrow \varphi_{ef,f}$$
$$S_{ef}$$

Any semigroup T of this form is said to be a *strong semilattice of semigroups* (of *type* $\mathcal{S}$ and *shape* L). In this case we refer to $L, (S_e), (\varphi_{f,e})$ as the *decomposition data* for T.

Example 2.4. Let $\mathcal{G}$ be the class of groups. Then a strong semilattice of semigroups of type $\mathcal{G}$, or more concisely a *strong semilattice of groups*, is called a *Clifford semigroup*. (One may also define Clifford semigroups in intrinsic terms, in which case the existence of suitable decomposition data for a given Clifford semigroup is a theorem rather than a tautology. This can be found in most reference texts on semigroups, e.g. [7, §IV.2].

If T is a strong semilattice of semigroups, with decomposition data L, (S_e) etc. then clearly each $\ell^1(S_e)$ is a subalgebra of $\ell^1(T)$, and in a loose sense the Banach algebra $\ell^1(T)$ is built out of the Banach algebras $\ell^1(S_e)$. This observation motivates the following definition, which is a mild generalisation of that in [2, Ch. 5].

The objects of study in this article are 'semilattice-shaped diagrams' in the category BAlg_1 of Banach algebras and contractive homomorphisms. An algebraist's definition (following the language of [9], say) might be as follows:

Provisional definition. A *strong semilattice of Banach algebras*, or a *semilattice diagram in* BAlg_1, is a pair (L, A) where L is a semilattice and A is a functor from the small category $(L, \succeq)$ to the category BAlg_1.

This definition relies on the interpretation of a poset as a special kind of small category: see [9, §1.2], for example. The 'proper' definition that follows is merely an unpacking of the terminology in the 'provisional' definition just given.

Definition 2.5. A *semilattice diagram in* BAlg_1 consists of: a semilattice L; a family $(A_e)_{e \in L}$ of Banach algebras; and *contractive* algebra homomorphisms $\phi_{f,e} : A_e \to A_f$ for each pair $(e, f) \in L \times L$ such that $e \succeq f$, which satisfy the following compatibility conditions:

- $\phi_{e,e}$ is the identity homomorphism on A_e;
- if $e \succeq f \succeq g$ in L then $\phi_{g,f} \circ \phi_{f,e} = \phi_{g,e}$.

We shall use the following notational shorthand: the expression 'let $(L, A) \in$ **SA**' is henceforth used as an abbreviation for the phrase 'let (L, A) be a semilattice diagram in BAlg_1', although on occasion we shall revert to the longer phrase in order to state certain results.

Remark. Note that in saying 'let (L, A) be a semilattice diagram ...' we are suppressing explicit mention of the family $(\phi_{f,e})$. This abuse of notation should not lead to confusion.

Remark. Our terminology perhaps deserves some remarks:

(a) One may think of such a pair (L, A) as a 'projective system of Banach algebras' indexed by the partially ordered set $(L, \succeq)$. However, such terminology does not seem apposite, as we shall consider neither the inductive nor projective limit of such a system, and since we later use properties of L that are not shared by arbitrary posets.

(b) The terminology '$(L, A) \in$ **SA**' strongly suggests that we might view strong semilattices of Banach algebras as objects of a category **SA**. While this is easy and not at all deep, we feel that such extra abstraction would not make the results of this paper any easier to follow.

Given $(L, A) \in$ **SA**, we can endow the ℓ^1-sum $\overset{(1)}{\bigoplus}_{e \in L} A_e$ with a multiplication that turns it into a Banach algebra. Before doing so, we shall introduce some basic notation regarding ℓ^1-sums: the point of our pedantry will be explained in due course.

Notation. Let $\mathbb{I}$ be an indexing set and let $(X_i)_{i \in \mathbb{I}}$ be a family of Banach spaces. We write $\mathfrak{X}$ for the ℓ^1-sum $\overset{(1)}{\bigoplus}_{i \in \mathbb{I}} X_i$, and for each $j \in \mathbb{I}$ we let ι_j^X denote the canonical inclusion map $X_j \hookrightarrow \mathfrak{X}$. If $x \in \coprod_{j \in \mathbb{I}} X_j$ then we shall sometimes write base(x) for the unique $i \in \mathbb{I}$ such that $x \in X_i$. Elements of the form $\iota_j^X(v)$, where $v \in X_j$, will be called *block elements*.

It is evident that two bounded linear maps with domain $\mathfrak{X}$ coincide if and only if they agree on all block elements; we shall use this repeatedly below without proof.

Definition 2.6. Let $(L, A) \in \mathbf{SA}$. We can form a Banach algebra $\mathfrak{alg}$, whose underlying vector space is the ℓ^1-sum $\bigoplus_{e \in L}^{(1)} A_e$ and whose multiplication is defined by the following rule:

$$\left(\sum_{e \in L} \iota_e^A a_e \right) \cdot \left(\sum_{f \in L} \iota_f^A b_f \right) := \sum_{g \in L} \left[\sum_{(e,f) \in L \times L \, : \, ef=g} \phi_{g,e}(a_e)\phi_{g,f}(b_f) \right]$$

We say that $\mathfrak{alg}$ is the *convolution algebra of* (L, A). If we need to make the dependence on A and L explicit we write $\mathfrak{alg}_{L,A}$.

Note that even if each A_e is unital, $\mathfrak{alg}$ need not be.

Simple examples. The following special cases of our construction should be kept in mind, and hopefully serve to clarify the general picture.

(a) Fix a Banach algebra $\mathfrak{B}$; let $A_e = \mathfrak{B}$ for all $e \in L$; and let each transition map $\phi_{f,e}$ be the identity map on $\mathfrak{B}$. Then there is an isomorphism $\ell^1(L) \widehat{\otimes} \mathfrak{B} \to \mathfrak{alg}_{L,A}$, defined by sending $e \widehat{\otimes} b$ to $\iota_e b$ for every $e \in L$ and $b \in \mathfrak{B}$.
   In particular, if $\mathfrak{B} = \mathbb{C}$ then $\mathfrak{alg} \cong \ell^1(L)$, the usual convolution algebra of the semigroup L.

(b) At the other extreme, suppose that $L = \{1\}$, the trivial semilattice containing only one element. Then clearly $\mathfrak{alg}_{L,A} \cong A_1$.

(c) Let $L = \{1, e\}$ be the two-element semilattice consisting of an identity element 1 and an idempotent e distinct from 1: then $1 \succeq e$. Let $\mathfrak{B}$ be any *unital* Banach algebra and define the functor $A : L \to \mathsf{BAlg}_1$ by taking $A_1 = \mathbb{C}$, $A_e = \mathfrak{B}$ and $\phi_{e,1} : \mathbb{C} \to \mathfrak{B}$ to be the homomorphism $\lambda \mapsto \lambda 1_{\mathfrak{B}}$.

$$\mathbb{C} \qquad\qquad \mathfrak{B}$$

$$1 \longrightarrow e$$

Then $\mathfrak{alg}_{L,A}$ is by definition isomorphic as a Banach space to the ℓ^1-sum $\mathbb{C} \oplus \mathfrak{B}$; and the multiplication on $\mathfrak{alg}_{L,A}$ is given by

$$(\lambda, b) \cdot (\mu, c) = (\lambda\mu, \lambda c + \mu b + bc) \qquad (\lambda, \mu \in \mathbb{C}; b, c \in \mathfrak{B}) \, .$$

Thus in this instance $\mathfrak{alg}_{L,A}$ is nothing but the forced unitisation of $\mathfrak{B}$.

The following example provides the main motivation for our definition of the convolution algebra.

Example 2.7 (ℓ^1-algebras of Clifford semigroups). Let L be a semilattice and let $(G_e)_{e \in L}$ be a semilattice diagram of groups. Let $G = \coprod_{e \in L} G_e$ be the corresponding semilattice of groups.

Applying the ℓ^1-group algebra functor to each G_e, we obtain a semi-lattice diagram in BAlg_1 with constituent Banach algebras $A_e = \ell^1(G_e)$. It is straightforward to check that $\mathfrak{alg}_{L,A}$ coincides with the convolution algebra $\ell^1(G)$.

$\mathfrak{alg}_{L,A}$ **as an** $\ell^1(L)$**-algebra.** The convolution algebras $\mathfrak{alg}_{L,A}$ have extra structure, whose precise formulation requires a digression into some of the general theory of Banach modules.

Definition 2.8. Recall that if K is a commutative, unital Banach algebra then a *Banach K-algebra* is a Banach algebra A, equipped with the structure of a symmetric Banach K-bimodule, such that for each $a \in A$ both left and right multiplication by a are K-module maps.

We say A is a *unit-linked* Banach K-algebra if it is unit-linked as a Banach K-bimodule.

Remark. Note that a unit-linked, Banach $\mathbb{C}$-algebra is nothing but a Banach algebra in the usual sense.

It is easily verified that $\mathfrak{alg}_{L,A}$ has the natural structure of a unit-linked Banach $\ell^1(L)$-algebra, as follows. We first define a symmetric action of $\ell^1(L)$ on the vector space $\bigoplus_{e \in L}^{(1)} A_e$ by

$$e \cdot \iota_i(a) = \iota_i(a) \cdot e := \iota_{ie}\phi_{ie,i}(a) \qquad (i, e \in L; a \in A_i).$$

Straightforward but tedious computations then show that for any $x \in \mathfrak{alg}_{L,A}$, the operators of left and right multiplication by x are both $\ell^1(L)$-bimodule maps.

Remark. If each A_e is a unital Banach algebra and each $\phi_{f,e}$ a unital homomorphism, then there is a canonical, isometric embedding of the vector space $\ell^1(L)$ into $\bigoplus_{e \in L}^{(1)} A_e$, defined by $e \mapsto \iota_e(1_{A_e})$. In this case, the way that we have defined multiplication in the convolution algebra $\mathfrak{alg}_{L,A}$ ensures that this isometric embedding is an algebra homomorphism from the convolution algebra $\ell^1(L)$ *into the centre of* $\mathfrak{alg}_{L,A}$.

It is clear that if the semilattice L has an identity $\mathbf{1}$, then $\iota_1(\mathbf{1})$ is an identity element for the Banach algebra $\mathfrak{alg}_{L,A}$; if this is the case then the $\ell^1(L)$-module strcuture on $\mathfrak{alg}_{L,A}$ is unit-linked. Later on it will be convenient, for technical reasons, to know that a stronger result is true.

Lemma 2.9. *Let* $(L, A) \in \mathbf{SA}$. *Suppose the convolution algebra* $\ell^1(L)$ *has an identity element* $u = \sum_{e \in L} \lambda_e e$; *then for every* $x \in \mathfrak{alg}_{L,A}$ *we have*

$$u \cdot x = x = x \cdot u \,.$$

Proof. By linearity and continuity, it suffices to show that $u \cdot \iota_f(a) = \iota_f(a) = \iota_f(a) \cdot u$ for any $f \in L$ and $a \in A_f$. Direct computation yields

$$u \cdot \iota_f(a) = \sum_{e \in L} \lambda_e e \cdot \iota_f(a)$$

$$= \sum_{e \in L} \lambda_e \iota_{ef} \phi_{ef,f}(a) \quad = \sum_{h \in fL} \left(\sum_{e \,:\, ef=h} \lambda_e \right) \iota_h \phi_{h,f}(a)$$

(manipulations with sums are justified since $\sum_e \lambda_e$ is an absolutely convergent series).

On the other hand, since u is an identity for $\ell^1(F)$ we have

$$f = u \cdot f = \sum_{e \in L} \lambda_e ef = \sum_{h \in fL} \left(\sum_{e \,:\, ef=h} \lambda_e \right) h$$

and by comparing coefficients of h on both sides of this equation we deduce that

$$\sum_{e \,:\, ef=h} \lambda_e = \begin{cases} 0 & \text{if } h \neq f \\ 1 & \text{if } h = f \end{cases}$$

Therefore $u \cdot \iota_f(a) = \iota_f \phi_{f,f}(a) = \iota_f(a)$. The proof that $\iota_f(a) \cdot u = \iota_f(a)$ is identical save for switching left and right multiplication, and we omit the details. $\square$

Multilinear extensions. Later on, we shall need to check whether certain functions defined on $\mathfrak{alg} \times \ldots \times \mathfrak{alg}^n$ are multilinear. In the cases we need it is fairly obvious whether this is the case, but to be precise we include the following lemma. First we set up some terminology.

Definition 2.10. Let $\mathbb{I}$ be an indexing set and let $(X_i)_{i \in \mathbb{I}}$ be a family of Banach spaces. Let E be a Banach space: if $\widetilde{T}$ is a function $\left(\coprod_{k \in \mathbb{I}} X_k \right)^n \to E$, an *extension of* $\widetilde{T}$ *to* $\mathfrak{X}$ is an n-linear function

$$T : \overbrace{\mathfrak{X} \times \ldots \times \mathfrak{X}}^{n} \to E$$

such that $T \circ (\iota_{k_1}, \ldots, \iota_{k_n}) = \widetilde{T}|_{X_{k_1} \times \ldots \times X_{k_n}}$ for all $k_1, \ldots, k_n \in \mathbb{I}$.

Lemma 2.11. *Let $n \in \mathbb{N}$ and let $C > 0$; let E be a Banach space and let*

$$\widetilde{T} : \left(\coprod_{i \in \mathbb{I}} X_i \right)^n \to E$$

be an arbitrary function. Then $\widetilde{T}$ has an extension to $\mathfrak{X}$ of norm $\leq C$, if and only if the following condition is satisfied:

(∗)   *for every n-tuple $(k_1, \ldots, k_n) \in \mathbb{I}^n$, the restriction of $\widetilde{T}$ to $X_{k_1} \times \ldots \times X_{k_n}$ is a bounded n-linear function with norm $\leq C$.*

Clearly if such an extension exists it will be unique, by linearity and continuity of the extension.

Proof of Lemma 2.11. If $\widetilde{T}$ extends to T and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{I}^n$, then for each $j \in \{1, \ldots, n\}$ and every $(n-1)$-tuple $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ where $x_r \in X_{k_r}$ for all r, the function $X_{j(r)} \to E$ given by

$$y \mapsto \widetilde{T}(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) = T\left(\iota_{k_1} x_1, \ldots, \iota_{k_j} y, \ldots, \iota_{k_n} x_n\right)$$

is bounded linear with norm $\leq \|T\|$. Hence $(*)$ holds.

Conversely, if $(*)$ holds we define the putative extension T as follows. If $u_1, \ldots, u_n \in \mathfrak{X}$ then each u_r has a unique representation as an absolutely convergent sum

$$u_r = \bigoplus_{k \in \mathbb{I}} \iota_k x_k^r \qquad (x_k^r \in X_k \ \text{ for all } k \in \mathbb{I})$$

and we make the definition

$$T(u_1, \ldots, u_n) := \sum_{\mathbf{k} \in \mathbb{I}^n} \widetilde{T}(x_{k_1}^1, \ldots, x_{k_n}^n)$$

where the sum on the right-hand side is absolutely convergent and thus well-defined; moreover, we have by condition $(*)$ the following estimate:

$$\sum_{\mathbf{k} \in \mathbb{I}^n} \|\widetilde{T}(x_{k_1}^1, \ldots, x_{k_n}^n)\| \leq C \sum_{\mathbf{k} \in \mathbb{I}^n} \|x_{k_1}^1\| \cdots \|x_{k_n}^n\| = C\|u_1\| \cdots \|u_n\| .$$

It remains only to show that T is indeed multilinear. Let $\lambda \in \mathbb{C}$ and $r \in \{1, \ldots, n\}$; let $v \in \mathfrak{X}$ have an expansion of the form $v = \sum_{k \in \mathbb{I}} \iota_k y_k$ where $\|v\| = \sum_{k \in \mathbb{I}} \|y_k\| < \infty$; then

$$
\begin{aligned}
& T(u_1, \ldots, \lambda u_r + v, \ldots, u_n) \\
&= \sum_{\mathbf{k} \in \mathbb{I}^n} \widetilde{T}(x_{k_1}^1, \ldots, \lambda x_{k_r}^r + y_{k_r}, \ldots, x^n k_n) \qquad \text{(by definition of } T) \\
&= \sum_{\mathbf{e} \in \mathbb{I}^n} \left[\begin{array}{l} \lambda \widetilde{T}(x_{k_1}^1, \ldots, x_{k_r}^r, \ldots, x_{k_n}^n) \\ + \widetilde{T}(x_{k_1}^1, \ldots, y_{k_r}, \ldots, x_{k_n}^n) \end{array} \right] \qquad \text{(by Condition } (*)) \\
&= \left\{ \begin{array}{l} \lambda T(u_1, \ldots, u_r, \ldots, u_n) \\ + T(u_1, \ldots, u_{r-1}, v, \ldots, u_n) \end{array} \right. \qquad \text{(definition of } T)
\end{aligned}
$$

and thus T is linear in the rth variable. Since r was arbitrary, T is multilinear as required. $\qquad \square$

3. HOCHSCHILD HOMOLOGY FOR $\mathfrak{alg}_{L,A}$

Having introduced the Banach algebras we shall study in the paper, we pause to give some general background on the homology theory that we will use.

The Hochschild *cohomology theory* of Banach algebras has been studied by many authors, and we refer the reader to [6] and [8] for the relevant

background. Hochschild homology has perhaps received less overt attention. Let us therefore briefly repeat the relevant definitions, in order to set out the notational conventions which will be used in this article.

Definition 3.1. Let A be a Banach algebra (not necessarily unital) and let M be a Banach A-bimodule. For $n \geq 0$ we define

$$\mathcal{C}_n(A, M) := M \mathbin{\widehat{\otimes}} A^{\widehat{\otimes} n}$$

where $\widehat{\otimes}$ denotes the projective tensor product of Banach spaces.

For $0 \leq i \leq n + 1$ the *face maps* $\partial_i^n : \mathcal{C}_{n+1}(A, M) \to \mathcal{C}_n(A, M)$ are the contractive linear maps given by

$$\partial_i^n(x \otimes a_1 \otimes \ldots \otimes a_{n+1}) = \begin{cases} x a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1} & \text{if } i = 0 \\ x \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1} & \text{if } 1 \leq i \leq n \\ a_{n+1} x \otimes a_1 \otimes \ldots \otimes a_n & \text{if } i = n + 1 \end{cases}$$

and the *Hochschild boundary operator* $\mathsf{d}_n : \mathcal{C}_{n+1}(A, M) \to \mathcal{C}_n(A, M)$ is given by

$$\mathsf{d}_n = \sum_{j=0}^{n+1} (-1)^j \partial_j^n \ .$$

With these definitions, the Banach spaces $\mathcal{C}_n(A, M)$ assemble into a chain complex

$$\ldots \xleftarrow{\ \mathsf{d}_{n-1}\ } \mathcal{C}_n(A, M) \xleftarrow{\ \mathsf{d}_n\ } \mathcal{C}_{n+1}(A, M) \xleftarrow{\ \mathsf{d}_{n+1}\ } \ldots$$

called the *Hochschild chain complex* of (A, M). We let

$$\mathcal{Z}_n(A, M) := \ker \mathsf{d}_{n-1} \qquad\qquad\qquad (\text{the space of } n\text{-}cycles)$$
$$\mathcal{B}_n(A, M) := \operatorname{im} \mathsf{d}_n \qquad\qquad\qquad (\text{the space of } n\text{-}boundaries)$$
$$\mathcal{H}_n(A, M) := \frac{\mathcal{Z}_n(A, M)}{\mathcal{B}_n(A, M)} \qquad\qquad (\text{the } n\text{th Hochschild homology group})$$

Remark. In the literature the spaces defined above are often referred to as the space of *bounded n-chains*, n-cycles and n-boundaries: the resulting homology groups are then called the *continuous* Hochschild homology groups of (A, M). In this article we have chosen to omit these adjectives for sake of brevity; this should not lead to any ambiguity.

Later, in Section 5, we shall also need the notions of K-*relative* homology groups where K is a unital Banach algebra. Let A be a unit-linked Banach K-algebra (see Definition 2.8).

Definition 3.2. Let X be a Banach A-bimodule. The space of K-*normalised* n-chains on A with coefficients in X is the Banach space $X \mathbin{\underset{K^e}{\widehat{\otimes}}} \left(A^{\widehat{\otimes}_K n} \right)$, and is denoted by $\mathcal{C}_n^K(A, X)$.

More explicitly, $\mathcal{C}_n^K(A, X)$ is defined to be the quotient of $X \mathbin{\widehat{\otimes}} A^{\widehat{\otimes} n}$ by the closed subspace $\mathcal{N}_n(K)$, where $\mathcal{N}_n$ is the closed linear span of all tensors of the form

$$xc \otimes a_1 \otimes \ldots \otimes a_n - x \otimes ca_1 \ldots \otimes a_n$$

$$\text{or } x \otimes a_1 \ldots \otimes a_j c \otimes a_{j+1} \otimes \ldots \otimes a_n - x \otimes a_1 \ldots \otimes a_j \otimes ca_{j+1} \otimes \ldots \otimes a_n \qquad (1 \leq j \leq n-1)$$

$$\text{or } x \otimes a_1 \otimes \ldots \otimes a_n c - cx \otimes a_1 \otimes \ldots \otimes a_n$$

where $x \in X$, $c \in K$ and $a_1, \ldots, a_n \in A$.

By considering the action of face maps on each such tensor, one sees that $\mathsf{d}_{n-1}(\mathcal{N}_n(K)) \subseteq \mathcal{N}_{n-1}(K)$, and so $\mathcal{N}_\bullet(K)$ is a subcomplex of the Hochschild chain complex $\mathcal{C}_*(A, X)$. Hence by a standard diagram chase the quotient spaces $\mathcal{C}_*^K(A, X)$ form a quotient complex of $\mathcal{C}_*(A, X)$, whose homology groups are the K-*relative homology groups* of A with coefficients in X.

The K-relative homology groups of A may be easier to compute than the 'full' homology groups, but it is not always clear how to relate the two families. Under certain circumstances the natural maps $\mathcal{H}_* \to \mathcal{H}_*^K$ are isomorphisms; we shall only require the following well-known instance of this phenomenon.

Definition 3.3. A Banach algebra K is said to be *contractible* if there exists $\Delta \in K \mathbin{\widehat{\otimes}} K$ satisfying the following conditions:

(i) $x \cdot \Delta = \Delta \cdot x$ for all $x \in K$;
(ii) $x\pi(\Delta) = x = \pi(\Delta)x$ for all $x \in K$

where $\pi : K \mathbin{\widehat{\otimes}} K \to K$ is the product map. Such a Δ, if it exists, is called a *diagonal for K*.

Let K be a unital, contractible algebra and $\mathcal{B}$ a unit-linked Banach K-algebra. As a special case of standard results, we can 'normalise chains on $\mathcal{B}$ with respect to K'. Later we will need a quantitative version of this fact, stated below.

Theorem 3.4. *Let K be a finite-dimensional, contractible Banach algebra with diagonal Δ. Then there is a sequence of positive reals $C_n > 0$ such that the following holds: whenever A is a unit-linked Banach K-algebra and X a Banach A-bimodule, there exists a chain map $\alpha : \mathcal{C}_*(A, X) \to \mathcal{C}_*(A, X)$ with the following properties:*

(a) *each α_n is K-normalised, i.e. factors through the quotient map $\mathcal{C}_n(A, X) \to \mathcal{C}_n^K(A, X)$;*
(b) *there exists a chain homotopy from id to α, given by bounded linear maps $t_n : \mathcal{C}_n(A, X) \to \mathcal{C}_{n+1}(A, X)$ such that $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ and $\|t_n\| \leq C_n$ for all n.*

(In particular, for each n the canonical map $\mathcal{H}_n(A, X) \to \mathcal{H}_n^K(A, X)$ is an isomorphism of seminormed spaces.)

We have stated this result in a simple version suitable for our future purposes, and not in its full generality. In particular it is given for chains and not for cochains as this is the setting in which we shall need it.

The crucial observation for us is that our splitting homotopy may be bounded independently of A and X; while the observation is far from new, it is not easy to find an explicit reference in the literature. A detailed proof may be found in Appendix A of the author's thesis [2].

Statement of the disintegration theorem. The aim of this article is to determine some of the cohomology groups for convolution algebras of the form $\mathfrak{alg}_{L,A}$. To make things clearer we adopt the following notation.

Notation. Let $(L, A) \in \mathbf{SA}$. We shall write

$$\begin{aligned}
\mathcal{C}_*[L; A] \text{ for } &\quad \mathcal{C}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\
\mathcal{Z}_*[L; A] \text{ for } &\quad \mathcal{Z}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\
\mathcal{B}_*[L; A] \text{ for } &\quad \mathcal{B}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\
\mathcal{H}_*[L; A] \text{ for } &\quad \mathcal{H}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A})
\end{aligned}$$

(This notation is meant to suggest that the simplicial chain complex of $\mathfrak{alg}_{L,A}$ may be defined more directly in terms of the pair (L, A), without introducing the intermediate object $\mathfrak{alg}_{L,A}$.)

In light of the main result in [3] that $\ell^1(L)$ has trivial simplicial homology for any semilattice L, it is natural to enquire if one can recover the homology groups $\mathcal{H}_*[L; A]$ from the family of homology groups $(\mathcal{H}_*(A_e, A_e))_{e \in L}$. To formulate this more precisely we make the following definition.

Definition 3.5. For each n we let $\mathcal{C}_n^{\mathrm{diag}}[L; A]$ denote the subspace

$$\overset{(1)}{\bigoplus}_{e \in L} \mathcal{C}_n(A_e, A_e) \quad \subseteq \quad \mathcal{C}_n[L; A]$$

It is clear that $\mathcal{C}_*^{\mathrm{diag}}[L; A]$ is a subcomplex of $\mathcal{C}_*[L; A]$. We shall sometimes refer to the elements of $\mathcal{C}_*^{\mathrm{diag}}[L; A]$ as *L-diagonal chains* on $\mathfrak{alg}_{L,A}$.

Now we are able to state the main result of this paper.

Theorem 3.6 (Disintegration theorem). *Let $(L, A) \in \mathbf{SA}$. Then the inclusion of $\mathcal{C}_*^{\mathrm{diag}}[L; A]$ into $\mathcal{C}_*[L; A]$ induces an isomorphism of homology groups.*

The point is, of course, that the complex $\mathcal{C}_*^{\mathrm{diag}}[L; A]$ depends only on the underlying set of L and the particular algebras $(A_e)_{e \in L}$, and *not* on the transition homomorphisms $A_e \to A_f$ which are used to construct $\mathfrak{alg}_{L,A}$.

Remark. The proofs of [1, Thm 2.1] and [5, Thm 4.6] may be easily extended to recover the cases $n = 1$ and $n = 2$ of Theorem 3.6. The novelty of our result lies not so much in the generalisation from Clifford semigroup algebras to strong semilattices of algebras, as in systematically solving the higher-degree cohomology problems.

We shall deduce Theorem 3.6 from a stronger but more technical one (Theorem 3.8 below), whose statement requires some more definitions. It turns out that the subcomplex $\mathcal{C}_*^{\mathrm{diag}}[L; A]$ is a chain summand in $\mathcal{C}_*[L; A]$, and we are able to choose a chain projection onto it with good properties.

Splitting off the diagonal part. Let $(L, A) \in \mathbf{SA}$, and denote the transition homomorphisms by $\phi_{f,e} : A_e \to A_f$ (where $f \preceq e$ in L). For each n, let $\mu_n^{L,A} : \mathcal{C}_n[L; A] \to \mathcal{C}_n^{\mathrm{diag}}[L; A]$ be the linear contraction defined by

$$\boxed{\mu_n^{L,A}(\iota_{e_0} a_0 \otimes \ldots \otimes \iota_{e_n} a_n) = \iota_p \phi_{p,e_0} a_0 \otimes \ldots \otimes \iota_p \phi_{p,e_n} a_n}$$

where $e_0, \ldots, e_n \in L$, $a_j \in A_{e_j}$ for all j and $p := e_0 \cdots e_n$. We shall occasionally drop the superscript and write μ_n when it is clear which (L, A) we are working with.

For each n, $\mu_n^{L,A}$ is a contractive, linear projection of $\mathcal{C}_n[L; A]$ onto $\mathcal{C}_n^{\mathrm{diag}}[L; A]$ (this is clear by computation on block elements).

Lemma 3.7. $\mu_*^{L,A}$ *is a chain map.*

Proof. It suffices to prove that $\mu_*^{L,A}$ commutes with each face map on the simplicial chain complex $\mathcal{C}_*[L; A]$.

We do this in full detail for the face map ∂_0. Let $n \geq 1$; let $e_0, \ldots, e_n \in L$; and let $a_j \in A_{e_j}$ for $j = 0, 1, \ldots, n$. We write p for $e_0 e_1 \cdots e_n$ and f for $e_0 e_1$; then

$$\mu_{n-1}^{L,A} \partial_0 \left(\iota_{e_0} a_0 \otimes \ldots \otimes \iota_{e_n} a_n \right)$$
$$= \mu_{n-1}^{L,A} \left((\iota_{e_0} a_0 \cdot \iota_{e_1} a_1) \otimes \ldots \otimes \iota_{e_n} a_n \right)$$
$$= \mu_{n-1}^{L,A} \left(\iota_f [\phi_{f,e_0}(a_0) \phi_{f,e_1}(a_1)] \otimes \ldots \otimes \iota_{e_n} a_n \right)$$
$$= \iota_p \left[\phi_{p,e_0}(a_0) \phi_{p,e_1}(a_1) \right] \otimes \ldots \otimes \iota_p \phi_{p,e_n} a_n \qquad \begin{array}{c} \text{(defn of } \mu_{n-1}^{L,A}, \text{ since} \\ f e_2 \cdots e_n = p) \end{array}$$
$$= (\iota_p \phi_{p,e_0} a_0) \cdot (\iota_p \phi_{p,e_1} a_1) \otimes \ldots \otimes \iota_p \phi_{p,e_n} a_n \qquad \begin{array}{c} \text{(since } \iota_p \text{ is a} \\ \text{homomorphism)} \end{array}$$
$$= \partial_0 \left(\iota_p \phi_{p,e_0} a_0 \otimes \ldots \otimes \iota_e \phi_{p,e_n} a_n \right)$$
$$= \partial_0 \mu_n^{L,A} \left(\iota_{e_0} a_0 \otimes \ldots \otimes \iota_{e_n} a_n \right) \qquad \qquad \text{(defn of } \mu_n^{L,A})$$

By linearity and continuity, we deduce that $\mu_{n-1}^{L,A} \partial_0 = \partial_0 \mu_n^{L,A}$ for all $n \geq 1$.

An exactly similar calculation shows that $\mu_{n-1}^{L,A} \partial_i = \partial_i \mu_n^{L,A}$, for each $i = 1, 2, \ldots, n$, and thus $\mu_{n-1}^{L,A} \mathsf{d}_{n-1} = \mathsf{d}_{n-1} \mu_n^{L,A}$ for all $n \geq 1$, as required. $\square$

Theorem 3.8. *Let* $\pi_n := \mathsf{id}_n - \mu_n : \mathcal{C}_n[L; A] \to \mathcal{C}_n[L; A]$. *Then the chain projection* $\pi_* : \mathcal{C}_*[L; A] \to \mathcal{C}_*[L; A]$ *is null-homotopic: that is, there exist bounded linear maps* $\sigma_n : \mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$ *for each* $n \geq 0$ *such that*

$$\mathsf{d}_n \sigma_n + \sigma_{n-1} \mathsf{d}_{n-1} = \pi_n \qquad (n \geq 1)$$

and $\mathsf{d}_0 \sigma_0 = \pi_0 = 0$.

The methods used to prove Theorem 3.8 are inspired by calculations in the papers [1, 5] for ℓ^1-algebras of Clifford semigroups; they constitute a more complicated implementation of the strategy used in the author's paper [3]. More precisely, as in [3] we combine two lines of attack: the naturality of π_* with respect to certain 'transfer maps' (to be defined below); and the fact that we can prove the theorem in the special case where L is a *finite, free* semilattice. We then set up an inductive argument to construct a splitting homotopy for π_*, by recursively 'transferring' known splitting formulas from the finite free case over to *natural* splitting formulas for the general case.

4. Transfer along semilattice homomorphisms

In this section we look at a particular class of homomorphisms between convolution algebras of the form $\mathfrak{alg}_{L,A}$, and at the chain maps thus induced between Hochschild chain complexes. We shall see later that these 'transfer maps' allow us to make precise certain 'changes of variable' that are required for later proofs.

Let $(L, A) \in \mathbf{SA}$. Suppose we have a semilattice H and a semigroup homomorphism $\alpha : H \to L$. Then we obtain a semilattice diagram $(H, A\alpha)$, by taking

$$(A\alpha)_e := A_{\alpha(e)} \qquad (e \in H)$$

and, whenever $e \succeq f$ in H, *defining* the transition homomorphism

$$(A\alpha)_e \xrightarrow{\phi_{f,e}^{A\alpha}} (A\alpha)_f$$

to be

$$A_{\alpha(e)} \xrightarrow{\phi_{\alpha f, \alpha e}^{A}} A_{\alpha(f)}.$$

We may thus form the convolution algebra $(H, A\alpha)$. There is a contractive linear map $\tau_\alpha : \mathfrak{alg}_{H,A\alpha} \to \mathfrak{alg}_{L,A}$, defined by

$$\tau_\alpha(\iota_e^{A\alpha} b) := \iota_{\alpha e}^{A} b \qquad (e \in H; b \in (A\alpha)_e).$$

(Strictly speaking, τ_α is defined by applying Lemma 2.11 to the function $\widetilde{\tau} : \coprod_{e \in H}(A\alpha)_e \to \mathfrak{alg}_{L,A}$, where $\widetilde{\tau}(b) = \iota_{\alpha e}^{A}(b)$ for $b \in (A\alpha)_e$.)

Claim. τ_α is an algebra homomorphism.

Proof. Since τ_α is linear and continuous it suffices to show that

$$(\dagger) \qquad \tau_\alpha(\iota_e^{A\alpha} b \cdot \iota_f^{A\alpha} c) = \tau_\alpha(\iota_e^{A\alpha} b) \cdot \tau_\alpha(\iota_f^{A\alpha} c)$$

for all $b, c \in \coprod_{e \in H}(A\alpha)_e$, where $e := \mathrm{base}(b)$ and $f := \mathrm{base}(c)$.

Since

$$\iota_e^{A\alpha} b \cdot \iota_f^{A\alpha} c \equiv \iota_{ef}^{A\alpha} \left[\phi_{ef,e}^{A\alpha}(b) \phi_{ef,f}^{A\alpha}(c) \right]$$

the left-hand side of ($\dagger$) is, by definition of τ_α,

$$\tau_\alpha \left(\iota_{ef}^{A\alpha} [\phi_{ef,e}^{A\alpha}(b) \phi_{ef,f}^{A\alpha}(c)] \right) = \iota_{\alpha(ef)}^{A} \left[\phi_{\alpha(ef),\alpha e}^{A}(b) \phi_{\alpha(ef),\alpha f}^{A}(c) \right] ;$$

while the right-hand side of (†) is, by the definitions of τ_α and of multiplication in $\mathfrak{alg}_{L,A}$,

$$\iota_{\alpha e}^A(b) \cdot \iota_{\alpha f}^A(c) = \iota_{\alpha(e)\alpha(f)}^A \left[\phi_{\alpha(e)\alpha(f),\alpha(e)}^A(b) \phi_{\alpha(e)\alpha(f),\alpha(f)}^A(c) \right] \ .$$

The desired identity (†) now follows, since $\alpha(ef) = \alpha(e)\alpha(f)$. $\qquad\square$

Definition 4.1. For $n \geq 0$, we let Tr_n^α denote the contractive linear map $\tau_\alpha^{\widehat{\otimes} n+1} : \mathcal{C}_n[H, A\alpha] \to \mathcal{C}_n[L, A]$. We call Tr_n^α the *transfer map along α* in degree n.

Note that since τ_α is a homomorphism, Tr_*^α is a chain map.

Notation. Later on, it will be useful to have the following shorthand: if $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ is an $(n+1)$-tuple of elements of $\coprod_{e \in L} A_e$, where $a_i \in A_{e(i)}$, say, then we write

$$\boldsymbol{\iota}_{\mathbf{e}}^A(\mathbf{a}) := \iota_{e_0}^A(a_0) \otimes \iota_{e_1}^A(a_1) \otimes \ldots \otimes \iota_{e_n}^A(a_n)$$

and write $\mathbf{e} = \mathrm{base}(\mathbf{a})$.

The next two lemmas are merely a matter of interpreting the new notation.

Lemma 4.2. *Let L, H, α, and A be as above. Then $\mathsf{Tr}^\alpha \boldsymbol{\iota}_{\mathbf{e}}^{A\alpha}(\mathbf{a}) = \boldsymbol{\iota}_{\alpha\mathbf{e}}^A(\mathbf{a})$ for every $e_0, \ldots, e_n \in L$ and $a_j \in A_{\alpha(e_j)}$, $j = 0, \ldots, n$.*

Lemma 4.3 ('Transfer is functorial'). *Let F, H, L be semilattices and let*

$$F \xrightarrow{\ \beta\ } H \xrightarrow{\ \alpha\ } L$$

be semigroup homomorhpisms. Then whenever $(L, A) \in \mathbf{SA}$,

$$\mathsf{Tr}^\alpha \mathsf{Tr}\, \boldsymbol{\iota}_\bullet^{A\alpha\beta} = \mathsf{Tr}^{\alpha\beta} \boldsymbol{\iota}_\bullet^{A\alpha\beta} = \boldsymbol{\iota}_{\alpha\beta\bullet}^A \quad .$$

Remark. There is a more general notion of transfer, where we not only allow ourselves to change the base semilattice but also the algebra structure above.

Specifically: let $(L, A), (H, B) \in \mathbf{SA}$ and let $\alpha : H \to L$ be a homomorphism of semilattices. A *transfer morphism* $(H, B) \to (L, A)$ is then given by a family $(\mu_h : B_h \to A_{\alpha h})_{h \in H}$ of contractive, unital algebra homomorphisms such that the diagram

$$
\begin{array}{ccc}
B_h & \xrightarrow{\ \mu_h\ } & A_{\alpha h} \\
{\scriptstyle \phi_{h,j}^B} \downarrow & & \downarrow {\scriptstyle \phi_{\alpha h, \alpha j}^A} \\
B_j & \xrightarrow[\ \mu_j\ ]{} & A_{\alpha j}
\end{array}
$$

commutes for every $h \leq j$ in H.

While this definition of transfer is in some sense 'the right one' from the category-theoretic viewpoint, it is not needed for what follows and we shall not pursue it further.

We now recall the chain projections $\mu^{L,A}$ that were defined earlier. It will be vital for the technical arguments used later that for fixed n, $\mu_n^{L,A}$ depends on (L,A) in a well-behaved way; and here 'well-behaved' means 'compatible with transfer' in the following sense.

Proposition 4.4 (μ is natural)**.** $\mu_n^\bullet$ *commutes with transfer in degree* n. *That is: given* $(L,A) \in \mathbf{SA}$, *and a semigroup homomorphism* $\alpha : H \to L$ *where* H *is a semilattice, we have a commuting diagram*

$$
\begin{array}{ccc}
\mathcal{C}_n[H; A\alpha] & \xrightarrow{\ \mu_n^{H,A\alpha}\ } & \mathcal{C}_n[H; A\alpha] \\[4pt]
\Big\downarrow{\scriptstyle \mathsf{Tr}_n^\alpha} & & \Big\downarrow{\scriptstyle \mathsf{Tr}_n^\alpha} \\[4pt]
\mathcal{C}_n[L; A] & \xrightarrow[\ \mu_n^{L,A}\ ]{} & \mathcal{C}_n[L; A]
\end{array}
$$

Proof. This is just book-keeping. Recall that Tr_n^α is just alternative notation for $\tau_\alpha^{\widehat{\otimes}\, n+1}$. Recall also that whenever $y \preceq x$ in H, $\phi_{y,x}^{A\alpha} = \phi_{\alpha y, \alpha x}^A$ by definition.

To keep the notation simple we shall give the proof in the special case $n = 2$: it should be clear from this how the general case works.

We shall show that $\mathsf{Tr}_2^\alpha\, \mu_2^{H,A\alpha}$ and $\mu_2^{L,A}\, \mathsf{Tr}_2^\alpha$ coincide on block elements of $\mathcal{C}_2[H; A\alpha]$: by linearity and continuity this will imply that they coincide on every element of $\mathcal{C}_2[H; A\alpha]$. Thus, let $e, f, g \in H$ and let $a \in (A\alpha)_e$, $b \in (A\alpha)_f$, $c \in (A\alpha)_g$. Let $p = efg$: then, since $\alpha(p) = \alpha(e)\alpha(f)\alpha(g)$,

$$
\begin{aligned}
\mathsf{Tr}_2^\alpha\, \mu_2^{H,A\alpha}(\iota_e^{A\alpha} a \otimes \iota_f^{A\alpha} b \otimes \iota_g^{A\alpha} c) &= \mathsf{Tr}_2^\alpha \left(\iota_p^{A\alpha} \phi_{p,e}^{A\alpha}(a) \otimes \iota_p^{A\alpha} \phi_{p,f}^{A\alpha}(b) \otimes \iota_p^{A\alpha} \phi_{p,g}^{A\alpha}(c) \right) \\
&= \tau_\alpha \iota_p^{A\alpha} \phi_{p,e}^{A\alpha}(a) \otimes \tau_\alpha \iota_p^{A\alpha} \phi_{p,f}^{A\alpha}(b) \otimes \tau_\alpha \iota_p^{A\alpha} \phi_{p,g}^{A\alpha}(c) \\
&= \iota_{\alpha p}^{A} \phi_{p,e}^{A\alpha}(a) \otimes \iota_{\alpha p}^{A} \phi_{p,f}^{A\alpha}(b) \otimes \iota_{\alpha p}^{A} \phi_{p,g}^{A\alpha}(c) \\
&= \iota_{\alpha p}^{A} \phi_{\alpha p,\alpha e}^{A}(a) \otimes \iota_{\alpha p}^{A} \phi_{\alpha p,\alpha f}^{A}(b) \otimes \iota_{\alpha p}^{A} \phi_{\alpha p,\alpha g}^{A}(c) \\
&= \mu_2^{L,A} \left(\iota_{\alpha e}^{A}(a) \otimes \iota_{\alpha f}^{A}(b) \otimes \iota_{\alpha g}^{A}(c) \right) \\
&= \mu_2^{L,A} \left(\tau_\alpha \iota_e^{A}(a) \otimes \tau_\alpha \iota_f^{A}(b) \otimes \tau_\alpha \iota_g^{A}(c) \right) \\
&= \mu_2^{L,A}\, \mathsf{Tr}_2^\alpha(\iota_e^{A\alpha} a \otimes \iota_f^{A\alpha} b \otimes \iota_g^{A\alpha} c)
\end{aligned}
$$

as required. $\qquad\square$

5. Explicit contractions in the finite free case

As in [3] a key role is played by the special case of a finite, free semilattice. Recall (Example 2.2) that if $\mathbb{I}$ is a finite set then the free semilattice generated by $\mathbb{I}$ is the set of all non-empty subsets of $\mathbb{I}$, with the product of two such subsets defined to be their union.

Proposition 5.1. *Let $\mathbb{I}$ be a finite set and F the free semilattice generated by $\mathbb{I}$; let $(F, B) \in \mathbf{SA}$. Let $\mu_*^{F,B} : \mathcal{C}_*[F; B] \to \mathcal{C}_*[F; B]$ be the chain map defined earlier. Then there exists a chain homotopy from $\mu^{F,B}$ to* id*: more precisely, there exist bounded linear maps $s_n^F : \mathcal{C}_n[F; B] \to \mathcal{C}_{n+1}[F; B]$ such that*

$$\mathsf{d}_n^{F,B} s_n^F + s_{n-1}^F \mathsf{d}_{n-1}^{F,B} = \mathsf{id}_n - \mu_n^{F,B}$$

and such that $\|s_n^F\|$ is bounded above by a constant depending only on F and n.

The rest of this section is given over to a somewhat indirect proof of the proposition above, and so may be skipped if the reader wishes to get straight to the core of the proof of Theorem 3.8.

Our approach will be to observe that one can identify the subcomplex of L-diagonal chains with the quotient complex of $\ell^1(L)$-normalised chains, and to exploit the fact that if F is a finite free semilattice then $\ell^1(F)$ is a contractible Banach algebra, to which we may apply Theorem 3.4.

L-diagonal representatives for $\ell^1(L)$-relative homology. We work in slightly greater generality than is strictly necessary, as this seems to clarify the ideas involved.

We recall from the discussion after Definition 2.8 that whenever $(L, A) \in \mathbf{SA}$, the convolution algebra $\mathfrak{alg}_{L,A}$ has the structure of an $\ell^1(L)$-algebra. Therefore we may consider chains and cochains which are normalised with respect to this action of $\ell^1(L)$. For sake of legibility we shall write $\mathcal{C}_n^L$ for the space of $\ell^1(L)$-normalised chains.

Lemma 5.2. *Let $(L, A) \in \mathbf{SA}$. For any $\mathbf{a} \in \left(\coprod_{e \in L} A_e\right)^{n+1}$, $\iota_{\mathbf{e}}\mathbf{a}$ and $\mu_n \iota_{\mathbf{e}}\mathbf{a}$ have the same image under the canonical quotient map $q : \mathcal{C}_*[L; A] \to \mathcal{C}_*^L[L; A]$.*

Proof. We shall show how the proof works for the case $n = 2$: the general case proceeds in the same way, but has to be formulated as a rather unwieldy inductive argument.

If x, y are elementary tensors in $\mathcal{C}_2[L; A]$, let us temporarily say that x and y are L-*equivalent* (denoted $x \sim y$) if $x - y \in \ker(q)$.

Let $\mathbf{a} = (a_0, a_1, a_2)$ be an arbitrary element of $\left(\coprod_{e \in L} A_e\right)^3$; let $e_j = \text{base}(a_j)$ for each j. Recall that for any $e, f \in L$ and $a \in A_e$

$$e \cdot \iota_e a = \iota_e a = (\iota_e a) \cdot e \qquad \text{and} \qquad f \cdot \iota_e a = ef \cdot \iota_e a = (\iota_e a) \cdot f$$

Let $p_2 = e_2$, $p_1 = e_1 e_2$ and $p = e_0 e_1 e_2$. By the definition of L-equivalence of tensors we see that

$$\begin{aligned}
\iota_{\mathbf{e}}\mathbf{a} &= \iota_{e_0} a_0 \otimes \iota_{e_1} a_1 \otimes \iota_{e_2} a_2 \\
&= \iota_{e_0} a_0 \otimes \iota_{e_1} a_1 \otimes \left[e_2 \cdot \iota_{e_2}(a_2)\right] \\
&\sim \iota_{e_0} a_0 \otimes \left[(\iota_{e_1} a_1) \cdot e_2\right] \otimes \iota_{e_2}(a_2) \\
&= \iota_{e_0} a_0 \otimes \left[p_1 \cdot \iota_{e_1} a_1\right] \otimes \left[p_2 \cdot \iota_{e_2}(a_2)\right]
\end{aligned}$$

and repeating this argument we obtain

$$\boldsymbol{\iota_e a} \sim \iota_{e_0} a_0 \otimes \left[p_1 \cdot \iota_{e_1} a_1 \right] \otimes \left[p_2 \cdot \iota_{e_2}(a_2) \right]$$
$$\sim \left[p \cdot \iota_{e_0} a_0 \right] \otimes \left[p_1 \cdot \iota_{e_1} a_1 \right] \otimes \left[p_2 \cdot \iota_{e_2}(a_2) \right]$$

We may now run this argument in the opposite direction, 'passing p from left to right'. Since $pp_j = p$ for $j = 0, 1, 2$, we get

$$\left[p \cdot \iota_{e_0} a_0 \right] \otimes \left[p_1 \cdot \iota_{e_1} a_1 \right] \otimes \left[p_2 \iota_{e_2}(a_2) \right]$$
$$\sim \left[p \cdot \iota_{e_0} a_0 \right] \otimes \left[p \cdot \iota_{e_1} a_1 \right] \otimes \left[p_2 \iota_{e_2}(a_2) \right]$$
$$\sim p\iota_{e_0} a_0 \otimes p\iota_{e_1} a_1 \otimes p\iota_{e_2} a_2$$
$$= \iota_p \phi_{p,e_0} a_0 \otimes \iota_p \phi_{p,e_1} a_1 \otimes \iota_p \phi_{p,e_2} a_2$$

and thus $\boldsymbol{\iota_e a} \sim \mu_n \boldsymbol{\iota_e a}$ as claimed.

$\square$

Constructing a homotopy for $\mathfrak{alg}_{F,B}$.

Lemma 5.3. *Let $\mathbb{I}$ be a finite set and F the free semilattice generated by $\mathbb{I}$. Then the Banach algebra $\ell^1(F)$ is finite-dimensional and contractible.*

This is well-known and we omit the proof, which can be found, for instance, as [2, Lemma 5.5.1]; see also [4, Example 1.6]. We note that [4] goes much further, providing an *algorithm* for computing the diagonal of an arbitrary finite semilattice.

Proposition 5.4. *Let F be as above and let $(F, B) \in \mathbf{SA}$; let $\mathcal{B}$ denote the convolution algebra $\mathfrak{alg}_{F,B}$; and let X be a Banach $\mathcal{B}$-bimodule. Then there exists a chain map $\alpha : \mathcal{C}_*(\mathcal{B}, X) \to \mathcal{C}_*(\mathcal{B}, X)$ with the following properties:*

(a) *each α_n factors through the quotient map $q_n : \mathcal{C}_n(\mathcal{B}, X) \to \mathcal{C}_n^F(\mathcal{B}, X)$;*
(b) *there exists a chain homotopy from id to α, given by bounded linear maps $t_n : \mathcal{C}_n(\mathcal{B}, X) \to \mathcal{C}_{n+1}(\mathcal{B}, X)$ satisfying $\mathsf{d}_n t_n + t_{n-1} \mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ for all n;*
(c) *the norm of each t_n is bounded above by some constant depending only on F and on n.*

Proof. By Lemma 5.3 the algebra $\ell^1(F)$ is contractible, so in particular has an identity element. By Lemma 2.9, the canonical inclusion of $\ell^1(F)$ into $\mathfrak{alg}_{F,B}$ sends the identity element of $\ell^1(F)$ to an identity element for $\mathfrak{alg}_{F,B}$, and the results (a)–(c) now follow as a special case of Theorem 3.4. $\square$

In particular, on taking $X = \mathcal{B}$ we see that every simplicial cycle on $\mathfrak{alg}_{F,B}$ is homologous to an F-diagonal one. However, this is not quite enough: we wish to show that every n-cycle $x \in \mathcal{Z}_n[F; B]$ is homologous to $\mu_n^{F,B} x$, so an extra step is needed.

This extra step can be done at a slightly more general level, and is given by the following trivial lemma. (It is stated for Hochschild homology but there is clearly a dual version for cohomology.)

Lemma 5.5 (Combining normalising projections). *Let A be a Banach algebra, X a Banach A-bimodule. Suppose we have two chain maps $\alpha, \lambda : \mathcal{C}_*(A, X) \to \mathcal{C}_*(A, X)$ such that*

- *α is chain-homotopic to the identity: i.e. there exist bounded linear maps $t_n : \mathcal{C}_n(A, X) \to \mathcal{C}_{n+1}(A, X)$ such that $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ for all n;*
- *$\lambda_* \alpha_* = \alpha_*$.*

Then λ_* is chain-homotopic to the identity: more precisely, we have

$$\mathsf{d}_n s_n + s_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \lambda_n$$

where $\|s_n\| \le (1 + \|\lambda_{n+1}\|)\|t_n\|$.

Proof. Let $n \ge 0$. We know that

$$\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n \ ;$$

composing on the left with λ_n on both sides, and recalling that λ is a chain map, we have

$$\mathsf{d}_n \lambda_{n+1} t_n + \lambda_n t_{n-1}\mathsf{d}_{n-1} = \lambda_n - \lambda_n \alpha_n = \lambda_n - \alpha_n$$

Subtracting this equation from the previous one yields

$$\mathsf{d}_n(\mathsf{id}_{n+1} - \lambda_{n+1})t_n + (\mathsf{id}_n - \lambda_n)t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \lambda_n \ ,$$

and on taking $s_n := (\mathsf{id}_{n+1} - \lambda_{n+1})t_n$ we have proved the lemma. $\qquad\square$

Proof of Proposition 5.1. By Proposition 5.4 there exists a chain map $\alpha : \mathcal{C}_*[F; B] \to \mathcal{C}_*[F, B]$ which factors through the quotient map $q : \mathcal{C}_*[F; B] \to \mathcal{C}_*^F[F; B]$ and which is homotopic to the identity. Since α factors through q_n, $\mu_n^{F,B}\alpha_n = \alpha_n$ for all n, by Lemma 5.2. Hence Lemma 5.5 applies. $\qquad\square$

Remark. Note that for fixed F the maps $s_n^{F,\text{—}}$ are natural in the second variable, in some sense ... but to make this precise, we need a more general notion of transfer.

6. Proving the main splitting theorem

Formulating the inductive step. The calculations of this section may seem rather messy. Figure 1 is meant to give some idea of the underlying strategy behind the details.

Definition 6.1. Let j be a non-negative integer and let

$$\left(\sigma_j^{L,A} : \mathcal{C}_j[L; A] \longrightarrow \mathcal{C}_{j+1}[L; A]\right)_{(L,A)\in\mathbf{SA}}$$

be a family of bounded linear maps. We consider the following four conditions that $\sigma^\bullet$ may or may not satisfy.

(R) For each $(L, A) \in \mathbf{SA}$,
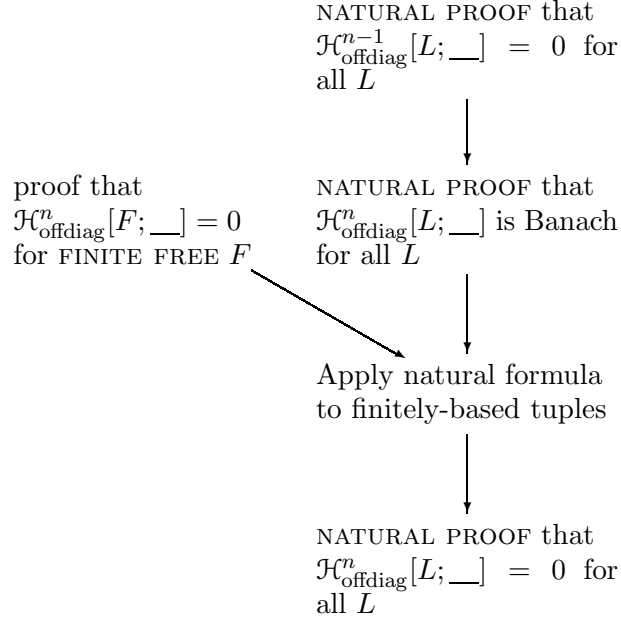
$$\pi_{j+1}^{L,A}\sigma_j^{L,A} = \sigma_j^{L,A} \ .$$

NATURAL PROOF that
$\mathcal{H}^{n-1}_{\text{offdiag}}[L; \underline{\quad}] \;=\; 0$ for
all L

proof that
$\mathcal{H}^n_{\text{offdiag}}[F; \underline{\quad}] = 0$
for FINITE FREE F

NATURAL PROOF that
$\mathcal{H}^n_{\text{offdiag}}[L; \underline{\quad}]$ is Banach
for all L

Apply natural formula
to finitely-based tuples

NATURAL PROOF that
$\mathcal{H}^n_{\text{offdiag}}[L; \underline{\quad}] \;=\; 0$ for
all L

FIGURE 1. The idea/motivation for the inductive step

(S) For each $(L, A) \in \mathbf{SA}$

$$\mathsf{d}_j^{L,A} \sigma_j^{L,A} \mathsf{d}_j^{L,A} = \mathsf{d}_j^{L,A} \pi_{j+1}^{L,A} \, .$$

(T) Whenever $(L, A) \in \mathbf{SA}$, H is a semilattice and $\alpha : H \to L$ is a
semigroup homomorphism, then the diagram

$$
\begin{CD}
\mathcal{C}_j[H; A\alpha] @>{\sigma_j^{H,A\alpha}}>> \mathcal{C}_{j+1}[H; A\alpha] \\
@V{\mathsf{Tr}_n^\alpha}VV @VV{\mathsf{Tr}_{n+1}^\alpha}V \\
\mathcal{C}_j[L; A] @>>{\sigma_j^{L,A}}> \mathcal{C}_{j+1}[L; A]
\end{CD}
$$

commutes.

(U) There is a constant $K_j > 0$ such that $\|\sigma_j^{L,A}\| \leq K_j$ for all $(L, A) \in \mathbf{SA}$.

Here (R) stands for *range fixed by* π; (S) for *semilattice-normalised splitting*; (T) for *transferable* map; and (U) for *uniformly bounded*.

Remark. Note that condition (R) tells us that our splitting map σ_j should take 'off-diagonal' cycles to 'off-diagonal' cycles.

Lemma 6.2 (Base for induction). *For each $(L, A) \in \mathbf{SA}$ let $\sigma_0^{L,A} : \mathcal{C}_0[L; A] \to \mathcal{C}_1[L; A]$ be the zero map. Then $\sigma_0^\bullet$ satisfies conditions* (R)–(U).

Proof. It is clear that taking $\sigma_0^{L,A} = 0$ for all $(L, A) \in \mathbf{SA}$ will satisfy conditions (R), (T) and (U). To show that condition (S) is satisfied we need only show that

$$0 = d_0^{L,A} \pi_1^{L,A} \qquad \text{for all } (L, A) \in \mathbf{SA}.$$

By linearity and continuity it suffices to verify this identity on block elements. So, let $e, f \in L$ and let $a \in A_e$, $b \in B_f$. Omitting superscripts for sake of clarity, we see that

$$\mu_1(\iota_e a \otimes \iota_f b) = \iota_{ef} \phi_{ef,e}(a) \otimes \iota_{ef} \phi_{ef,f}(b) \equiv (f \cdot \iota_e a) \otimes (e \cdot \iota_f b)$$

and so

$$\begin{aligned}
\mathsf{d}_0 \mu_1(\iota_e a \otimes \iota_f b) &= \mathsf{d}_0 \left((f \cdot \iota_e a) \otimes (e \cdot \iota_f b) \right) \\
&= (\iota_f b \cdot \iota_e a) - (\iota_e a \cdot \iota_f b) \quad = \mathsf{d}_0(\iota_e a \otimes \iota_f b)
\end{aligned}$$

which gives us $\mathsf{d}_0 \pi_1(\iota_e a \otimes \iota_f b) = \mathsf{d}_0(\iota_e a \otimes \iota_f b) - \mathsf{d}_0 \mu_1(\iota_e a \otimes \iota_f b) = 0$ as required. $\qquad \square$

Remark. It may help to think of the above proof as the 'predual version' of the following statement: *every derivation* $\mathfrak{alg}_{L,A} \to (\mathfrak{alg}_{L,A})'$ *is automatically* L-*normalised.*

The following proposition is our inductive step, and is the heart of the main splitting theorem.

Proposition 6.3. *Let $n \geq 1$, and suppose that there exists a family of linear maps* $\sigma_{n-1}^{L,A} : \mathcal{C}_{n-1}[L; A] \to \mathcal{C}_n[L; A]$ *which satisfies conditions* (R), (S), (T) *and* (U).

Then there exists a family of linear maps $\sigma_n^{L,A} : \mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$ *which satisfies conditions* (R), (T) *and* (U)*, and also satisfies*

$$(\spadesuit) \qquad d_n^{L,A} \sigma_n^{L,A} + \sigma_{n-1}^{L,A} d_{n-1}^{L,A} = \pi_n^{L,A} \qquad \text{for all } (L, A) \in \mathbf{SA} .$$

Let us first see how this proposition gives Theorem 3.8.

Proof of Theorem 3.8, using Proposition 6.3. We shall prove a stronger statement, namely that there exists a family of bounded linear maps $\sigma_n^{L,A} :$ $\mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$ for each $n \geq 0$ and $(L, A) \in \mathbf{SA}$, such that

$$\mathsf{d}_n^{L,A} \sigma_n^{L,A} + \sigma_{n-1}^{L,A} \mathsf{d}_{n-1}^{L,A} = \pi_n^{L,A} \qquad (n \geq 1) \qquad \text{and} \qquad \mathsf{d}_0^{L,A} \sigma_0^{L,A} = 0$$

for all $(L, A) \in \mathbf{SA}$, *and* such that for each $n \in \mathbb{Z}_+$ the family $(\sigma_n^{L,A})_{(L,A) \in \mathbf{SA}}$ satisfies conditions (R), (T) and (U).

The proof is by strong induction on n.

We have seen (Lemma 6.2) that taking $\sigma_0^\bullet = 0$ gives a family $(\sigma_0^{L,A})_{(L,A) \in \mathbf{SA}}$ which satisfies conditions (R), (S), (T) and (U), and which satisfies

$$\mathsf{d}_0^{L,A} \sigma_0^{L,A} = 0 .$$

Therefore by Proposition 6.3 there exists a family $(\sigma_1^{L,A})_{(L,A)\in\mathbf{SA}}$ which satisfies conditions (R), (T) and (U), and such that

$$\mathsf{d}_1^{L,A}\sigma_1^{L,A} + \sigma_0^{L,A}\mathsf{d}_0^{L,A} = \pi_1^{L,A}$$

for all $(L,A) \in \mathbf{SA}$. Hence

$$\mathsf{d}_1^{L,A}\sigma_1^{L,A}\mathsf{d}_1^{L,A} = \left(\mathsf{d}_1^{L,A}\sigma_1^{L,A} + \sigma_0^{L,A}\mathsf{d}_0^{L,A}\right)\mathsf{d}_1^{L,A} = \pi_1^{L,A}\mathsf{d}_1^{L,A} = \mathsf{d}_1^{L,A}\pi_2^{L,A}$$

so that $\sigma_1^\bullet$ also satisfies condition (S).

Now suppose that there exists $n \geq 2$ and families $\sigma_{n-2}^\bullet$, $\sigma_{n-1}^\bullet$ which both satisfy conditions (R), (T) and (U), and which also satisfy

$$\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A} + \sigma_{n-2}^{L,A}\mathsf{d}_{n-2}^{L,A} = \pi_{n-1}^{L,A}$$

for all $(L,A) \in \mathbf{SA}$. Then

$$\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \left(\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A} + \sigma_{n-2}^{L,A}\mathsf{d}_{n-2}^{L,A}\right)\mathsf{d}_{n-1}^{L,A} = \pi_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \mathsf{d}_{n-1}^{L,A}\pi_n^{L,A}$$

and so $\sigma_{n-1}^\bullet$ satisfies condition (S). Hence by Proposition 6.3 there exists a family $(\sigma_n^{L,A})_{(L,A)\in\mathbf{SA}}$ which satisfies conditions (R), (T) and (U) and which also satisfies

$$\mathsf{d}_n^{L,A}\sigma_n^{L,A} + \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \pi_n^{L,A} \quad .$$

This completes the induction, and hence Theorem 3.8 is proved *assuming that Proposition 6.3 is valid.* $\qquad\square$

The proof of Proposition 6.3 occupies the rest of this section.

Proof of the inductive step (Proposition 6.3). *Throughout the rest of this section we fix $n \geq 1$ and let F denote the finite semilattice freely generated by $n+1$ idempotents $f_0, \ldots, f_n$.*

Let $(F,T) \in \mathbf{SA}$. By Proposition 5.1 there exist bounded linear maps s_n^T and s_{n-1}^T such that:

$$(\diamondsuit) \qquad \boxed{\mathsf{d}_n^{F,T}s_n^T = \pi_n^{F,T} - s_{n-1}^T\mathsf{d}_{n-1}^{F,T}}$$

Recall also that $\|s_n^T\| \leq C_n$ for some constant C_n that is independent of T.

The formula $(\diamondsuit)$ says that every n-chain z in the image of $\pi_n^{F,T}$ can be written as the sum of an n-boundary and a chain which is a linear image of $\mathsf{d}_{n-1}^{F,T}z$. The next step is aimed, roughly speaking, at improving this decomposition so that z is the sum of an n-boundary and a chain which is the linear image of $\mathsf{d}_{n-1}^{F,T}z$ *under a natural map* (Equation $(\heartsuit)$ below).

Since $\pi^{F,T}$ is a chain projection, $(\diamondsuit)$ implies that

$$(\dagger\dagger) \qquad \mathsf{d}_n^{F,T}\pi_{n+1}^{F,T}s_n^T = \pi_n^{F,T}\mathsf{d}_n^{F,T}s_n^T = \pi_n^{F,T} - \pi_n^{F,T}s_{n-1}^T\mathsf{d}_{n-1}^{F,T} ;$$

but, since condition (S) holds for $\sigma_{n-1}^\bullet$, we also know that

$$\mathsf{d}_{n-1}^{F,T}\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right) = 0 .$$

Hence post-multiplying on both sides of (††) by $\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}$ gives us

$$\mathsf{d}_n^{F,T}\pi_{n+1}^{F,T}s_n^T\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)$$

$$= \pi_n^{F,T}\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)$$

$$= \pi_n^{F,T} - \pi_n^{F,T}\sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T} \qquad \text{(since } \pi_n \text{ is a projection)}$$

$$= \pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T} \qquad \text{(since (R) holds for } \sigma_{n-1}^\bullet\text{)}$$

Before proceeding we introduce some auxiliary notation to ease the congestion of indices: let $\psi_T : \mathcal{C}_n[F;T] \to \mathcal{C}_n[F;T]$ denote the *bounded linear map*

$$\psi_T := \pi_{n+1}^{F,T}s_n^T\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)$$

This allows us to rewrite the rather cumbersome formula above more concisely, as

($\heartsuit$) $\qquad\qquad \boxed{\mathsf{d}_n^{F,T}\psi_T = \pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}}$

Observe for later reference that $\pi_{n+1}^{F,T}\psi_T = \psi_T$. Also, since condition (U) is assumed to hold, there is a constant K_{n-1} such that $\|\sigma_{n-1}^{F,T}\| \le K_{n-1}$. Therefore

$$\|\psi_T\| \le \|s_n^T\|(1 + K_{n-1}\|\mathsf{d}_{n-1}^{F,T}\|) \le C_n(1 + (n+1)K_{n-1})$$

where the right-hand side depends only on n.

Now let $(L, A) \in \mathbf{SA}$. Let $x_0, \ldots, x_n \in L$, and observe that there is a unique, well-defined homomorphism of semilattices $F \to L$ which sends $f_j \mapsto x_j$ for $j = 0, \ldots, n$. We denote this homomorphism by $\widehat{\mathbf{x}} : F \to L$ — the notation is meant to suggest that we think of $\widehat{\mathbf{x}}$ as 'evaluation' of the 'free variables' f_j at particular values x_j — and write $A\widehat{\mathbf{x}}$ for $A_\bullet \circ \widehat{\mathbf{x}} : F \to \mathsf{BAlg}_1$.

Since we have a homomorphism $\widehat{\mathbf{x}} : F \to L$, we are in a position to bring in the transfer maps from Section 4. Consider the semilattice diagram in BAlg_1 given by $(F, A\widehat{\mathbf{x}})$, and the associated transfer chain map

$$\mathcal{C}_*[F; A\widehat{\mathbf{x}}] \xrightarrow{\mathsf{Tr}_*^{\widehat{\mathbf{x}}}} \mathcal{C}_*[L; A]$$

Now suppose that for $j = 0, 1\ldots, n$ we have $a_j \in (A\widehat{\mathbf{x}})_{f_j} = A_{x_j}$, so that $\iota_{\mathbf{x}}^A$ is an element of $\mathcal{C}_n[L; A]$. We shall define $\sigma_n^{L,A}$ on elements of this form and extend using Lemma 2.11.

Intuitively, Equation ($\heartsuit$) is a kind of formal identity in the unknowns $f_0, \ldots, f_n$ and $a_0, \ldots, a_n$, subject to the f_j being commuting idempotents; therefore equality must be preserved when we 'evaluate' each f_j at x_j. With this observation in mind, we define a function $\widetilde{\sigma}_n^{L,A} : \left(\coprod_{x\in L} A_x\right)^{n+1} \to \mathcal{C}_{n+1}[L; A]$ as follows: given $\mathbf{a} \in \left(\coprod_{x\in L} A_x\right)^{n+1}$, let $\mathbf{e} = \mathrm{base}(\mathbf{a})$ and set

$$\boxed{\widetilde{\sigma}_n^{L,A}(\mathbf{a}) := \mathsf{Tr}_{n+1}^{\widehat{\mathbf{e}}}\,\psi_{A\widehat{\mathbf{e}}}\iota_{\mathbf{f}}^{A\widehat{\mathbf{e}}}(\mathbf{a})}$$

We claim that $\widetilde{\sigma}_n^{L,A}$ has a bounded linear extension to $\mathfrak{alg}_{L,A}$. To see this, recall from Lemma 2.11 that $\widetilde{\sigma}_n^{L,A}$ has a linear extension of norm $\leq K$ if and only if, for every $\mathbf{x} = (x_0, \ldots, x_n) \in L^{n+1}$, the restriction of $\widetilde{\sigma}_n^{L,A}$ to $A_{x_0} \times \ldots \times A_{x_n}$ is multilinear with norm $\leq K$. But by the way we have defined $\widetilde{\sigma}_n^{L,A}$, for fixed $\mathbf{x} = (x_0, \ldots, x_n) \in L^{n+1}$ this restriction coincides with $\mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}} \psi_{A\widehat{\mathbf{x}}} \boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}$, which is clearly multilinear since $\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}$ is multilinear and $\mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}} \psi_{A\widehat{\mathbf{x}}}$ is linear. Moreover, for any $(a_0, \ldots, a_n)$ we have

$$\left\| \widetilde{\sigma}_n^{L,A} \boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(a_0, \ldots, a_n) \right\| \leq \|\psi_{A\widehat{\mathbf{x}}}\| \, \|a_0\| \ldots \|a_n\|$$

and we saw earlier that $\|\psi_{A\widehat{\mathbf{x}}}\| \leq K_n$ for some constant K_n that depends only on n. Hence by Lemma 2.11 $\widetilde{\sigma}_n^{L,A}$ has a bounded $(n+1)$-linear extension to $\mathfrak{alg}_{L,A}$, as claimed. This extension may be canonically identified with a bounded *linear* map $\mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$, which we denote by $\sigma_n^{L,A}$.

At this point, we have constructed for each $(L, A) \in \mathbf{SA}$ a bounded linear map $\sigma_n^{L,A} : \mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$, which satisfies $\|\sigma_n^{L,A}\| \leq K_n$ for some constant K_n independent of L and A. So in particular our family $\sigma_n^{\bullet}$ satisfies condition (U).

Next, we shall prove that the family $\sigma_n^{\bullet}$ satisfies condition (T). To do this we must show that the diagram

$$
\begin{array}{ccc}
\mathcal{C}_n[H; A\alpha] & \xrightarrow{\;\;\sigma_n^{H,A\alpha}\;\;} & \mathcal{C}_{n+1}[H; A\alpha] \\
{\scriptstyle \mathsf{Tr}_n^{\alpha}} \downarrow & & \downarrow {\scriptstyle \mathsf{Tr}_{n+1}^{\alpha}} \\
\mathcal{C}_n[L; A] & \xrightarrow[\;\;\sigma_n^{L,A}\;\;]{} & \mathcal{C}_{n+1}[L; A]
\end{array}
$$

commutes whenever we have a semilattice homomorphism $H \xrightarrow{\;\alpha\;} L$.

By linearity and continuity it suffices to check this on block elements of $\mathcal{C}_n[H; A\alpha]$. Let $y_0, \ldots, y_{n+1} \in H$: for each j, let x_j denote $\alpha(y_j) \in L$, and let $a_j \in (A\alpha)_{y_j} = A_{x_j}$. Recall that by Lemma 4.3

$$\mathsf{Tr}_{n+1}^{\alpha} \, \mathsf{Tr}_{n+1}^{\widehat{\mathbf{y}}} = \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}$$

and

$$\mathsf{Tr}_n^{\alpha} \, \boldsymbol{\iota}_{\mathbf{y}}^{A\alpha}(\mathbf{a}) = \boldsymbol{\iota}_{\mathbf{x}}^{A}(\mathbf{a}) \; .$$

Therefore,

$$\mathsf{Tr}_{n+1}^{\alpha} \, \sigma_n^{H,A\alpha} \boldsymbol{\iota}_{\mathbf{y}}^{A\alpha}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\alpha} \, \mathsf{Tr}_{n+1}^{\widehat{\mathbf{y}}} \psi_{A\alpha\mathbf{y}} \boldsymbol{\iota}_{\mathbf{f}}^{A\alpha\widehat{\mathbf{y}}}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\alpha\widehat{\mathbf{y}}} \psi_{A\alpha\widehat{\mathbf{y}}} \boldsymbol{\iota}_{\mathbf{f}}^{A\alpha\widehat{\mathbf{y}}}(\mathbf{a})$$

and

$$\sigma_n^{L,A} \, \mathsf{Tr}_n^{\alpha} \, \boldsymbol{\iota}_{\mathbf{y}}^{A\alpha}(\mathbf{a}) = \sigma_n^{L,A} \boldsymbol{\iota}_{\mathbf{x}}^{A}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}} \psi_{A\widehat{\mathbf{x}}} \boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a})$$

Since $\alpha\widehat{\mathbf{y}} = \widehat{\mathbf{x}}$, the previous two equations combine to give

$$\mathsf{Tr}_{n+1}^{\alpha} \, \sigma_n^{H,B} \boldsymbol{\iota}_{\mathbf{y}}^{A\alpha}(\mathbf{a}) = \sigma_n^{L,A} \, \mathsf{Tr}_n^{\alpha} \, \boldsymbol{\iota}_{\mathbf{y}}^{A\alpha}(\mathbf{a})$$

as required.

$$\mathcal{C}_{n-1}[F; A\widehat{\mathbf{x}}] \xleftarrow[\sigma_{n-1}^{F,A\widehat{\mathbf{x}}}]{\mathsf{d}_{n-1}^{F,A\widehat{\mathbf{x}}}} \mathcal{C}_n[F; A\widehat{\mathbf{x}}] \xleftarrow{\mathsf{d}_n^{F,A\widehat{\mathbf{x}}}} \mathcal{C}_{n+1}[F; A\widehat{\mathbf{x}}]$$

$$\mathsf{Tr}_{n-1}^{\widehat{\mathbf{x}}} \Big\downarrow \qquad\qquad \mathsf{Tr}_n^{\widehat{\mathbf{x}}} \Big\downarrow$$

$$\mathcal{C}_{n-1}[L; A] \xleftarrow[\sigma_{n-1}^{L,A}]{\mathsf{d}_{n-1}^{L,A}} \mathcal{C}_n[L; A] \xdashleftarrow{\mathsf{d}_n^{L,A}} \mathcal{C}_{n+1}[L; A]$$

Figure 2. Transferring from F to L

It remains only to verify conditions (R) and $(\spadesuit)$. Let $(L, A) \in \mathbf{SA}$: we must show that $\pi_{n+1}^{L,A}\sigma_n = \sigma_n^{L,A}$ and $\mathsf{d}_n^{L,A}\sigma_n^{L,A} = \pi_n^{L,A} - \sigma_{n-1}^{L,A}\mathsf{d}^{L,A}$.

Since we now know that $\sigma_n^{L,A}$ is a bounded multilinear map $\mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$, it suffices by linearity and continuity to check both putative identities on block elements of the form $\boldsymbol{\iota}_{\mathbf{x}}^A\mathbf{a}$.

This is now mere diagram-chasing, given the machinery set up earlier. The diagram shown in Figure 2 is worth keeping in mind. Since

$$\begin{aligned}
\pi_{n+1}^{L,A}\sigma_n^{L,A}(\boldsymbol{\iota}_{\mathbf{x}}^A\mathbf{a}) &= \pi_{n+1}^{L,A}\,\mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\,\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a}) && \text{(by definition)} \\
&= \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\,\pi^{F,A\widehat{\mathbf{x}}}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a}) && (\pi \text{ commutes with transfer}) \\
&= \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\,\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a}) && (\text{see remark after } (\heartsuit)) \\
&= \sigma_n^{L,A}(\boldsymbol{\iota}_{\mathbf{x}}^A\mathbf{a})
\end{aligned}$$

condition (R) is satisfied. Finally (and *this* is where we make use of condition (T) in our induction):

$$\begin{aligned}
&\mathsf{d}_n^{L,A}\sigma_n^{L,A}(\boldsymbol{\iota}_{\mathbf{x}}^A\mathbf{a}) \\
&= \mathsf{d}_n^{L,A}\,\mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\,\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a}) && \text{(by definition)} \\
&= \mathsf{Tr}_n^{\widehat{\mathbf{x}}}\,\mathsf{d}_n^{F,A\widehat{\mathbf{x}}}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a}) && \text{(transfer is a chain map)} \\
&= \mathsf{Tr}_n^{\widehat{\mathbf{x}}}\left(\pi_n^{F,A\widehat{\mathbf{x}}} - \sigma_{n-1}^{F,A\widehat{\mathbf{x}}}\mathsf{d}_{n-1}^{F,A\widehat{\mathbf{x}}}\right)(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) && \text{(by the 'formal identity' } (\heartsuit)) \\
&= \left(\mathsf{Tr}_n^{\widehat{\mathbf{x}}}\,\pi_n^{F,A\widehat{\mathbf{x}}} - \mathsf{Tr}_n^{\widehat{\mathbf{x}}}\,\sigma_{n-1}^{F,A\widehat{\mathbf{x}}}\mathsf{d}_{n-1}^{F,A\widehat{\mathbf{x}}}\right)(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) \\
&= \left(\mathsf{Tr}_n^{\widehat{\mathbf{x}}}\,\pi_n^{F,A\widehat{\mathbf{x}}} - \sigma_{n-1}^{L,A}\,\mathsf{Tr}_{n-1}^{\widehat{\mathbf{x}}}\,\mathsf{d}_{n-1}^{F,A\widehat{\mathbf{x}}}\right)(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) && (\text{condition } (\mathsf{T}) \text{ for } \sigma_{n-1}^{\bullet}) \\
&= \left(\pi_n^{L,A}\,\mathsf{Tr}_n^{\widehat{\mathbf{x}}} - \sigma_{n-1}^{L,A}\,\mathsf{Tr}_{n-1}^{\widehat{\mathbf{x}}}\,\mathsf{d}_{n-1}^{F,A\widehat{\mathbf{x}}}\right)(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) && (\pi \text{ commutes with transfer}) \\
&= \left(\pi_n^{L,A}\,\mathsf{Tr}_n^{\widehat{\mathbf{x}}} - \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A}\,\mathsf{Tr}_n^{\widehat{\mathbf{x}}}\right)(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) && \text{(transfer is a chain map)} \\
&= \left(\pi_n^{L,A} - \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A}\right)\mathsf{Tr}_n^{\widehat{\mathbf{x}}}(\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}\mathbf{a}) \\
&= \left(\pi_n^{L,A} - \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A}\right)(\boldsymbol{\iota}_{\mathbf{x}}^A\mathbf{a})
\end{aligned}$$

where the last step follows from the definition of $\widehat{\mathbf{x}}$ and Lemma 4.2.

Thus the family $(\sigma_n^{L,A})_{(L,A)\in\mathbf{SA}}$ satisfies Equation ($\spadesuit$), and this concludes the proof of Proposition 6.3. In view of the previous section, this completes the proof of Theorem 3.8 and hence of Theorem 3.6.

7. Applications to ℓ^1-convolution algebras

Clifford semigroups. Bowling and Duncan observed [1, Thm 2.1] that the ℓ^1-algebra of *any* Clifford semigroup S is weakly amenable. By using Theorem 3.6 we can extend their result to higher-degree cohomology if we make further assumptions about S.

Proposition 7.1. *Let $S = \coprod_{e\in L} G_e$ be a Clifford semigroup over the semilattice L. Suppose that each G_e is amenable. Then $\mathcal{H}_n(\ell^1(S), \ell^1(S)) = 0$ and $\mathcal{H}^n(\ell^1(S), \ell^1(S)') = 0$ for all $n \geq 1$.*

Proof. If G_e is amenable then $\ell^1(G_e)$ is amenable with constant 1, so that the simplicial chain complex $\mathcal{C}_*(\ell^1(G_e, \ell^1(G_e))$ is weakly split in degrees 1 and above with constants independent of e. Since

$$\mathcal{C}_n^{\mathrm{diag}}(\ell^1(S), \ell^1(S)) \overset{(1)}{=} \bigoplus_{e\in L} \mathcal{C}_n(\ell^1(G_e), \ell^1(G_e))$$

it follows that $\mathcal{H}_n^{\mathrm{diag}}(\ell^1(S), \ell^1(S)) = 0$ for all $n \geq 1$. Hence by Theorem 3.6,

$$\mathcal{H}_n(\ell^1(S), \ell^1(S)) = \mathcal{H}_n^{\mathrm{diag}}(\ell^1(S), \ell^1(S)) = 0 \qquad \text{for all } n \geq 1$$

as required. The statement about cohomology follows by a standard duality argument, see e.g. [8, Coroll. 1.3]. $\square$

Remark. Note that in the degenerate case where $G_e = \{1\}$ for each e, we recover the main result of [3], namely the simplicial triviality of the convolution algebra of a semilattice.

In [3] vanishing results for the simplicial homology of ℓ^1-semilattice algebras were used to deduce vanishing results for cohomology with arbitrary symmetric coefficients. This procedure carries over to the case of commutative Clifford semigroups.

Theorem 7.2. *Let S be a commutative Clifford semigroup and let X be any symmetric Banach $\ell^1(S)$-bimodule. Then $\mathcal{H}^n(\ell^1(S), X) = 0$ for all $n \geq 1$.*

Proof. We first note that $\mathcal{H}^n(\ell^1(S), X) \cong \mathcal{H}^n(\ell^1(S)^{\#}, X_1)$ where X_1 has underlying Banach space X and is the natural *unital* bimodule induced from X. (This is a special case of a general result on cohomology of unitisations: see [6, Exercise III.4.10] or [8, Ch. 1] for details.)

Clearly $\ell^1(S)^{\#} = \ell^1(S^{\#})$ where $S^{\#}$ is itself a (unital) commutative Clifford semigroup. By Proposition 7.1, $\ell^1(S^{\#})$ is simplicially trivial; hence the conditions of [3, Thm 3.2] are satisfied and we may use that theorem to deduce that $\mathcal{H}^n(\ell^1(S)^{\#}, X_1) = 0$ for all $n \geq 1$, as required. $\square$

Remark. Note that the case $n = 1$ follows from [1, Thm 2.1] and the well-known fact that weak amenability for commutative Banach algebras forces all bounded derivations with symmetric coefficients to vanish.

What can be said for more general coefficients? Here matters are more delicate, and one difficulty seems to be the lack of a chain projection onto the subcomplex of L-diagonal chains (recall that in the case of *simplicial* chains we could exploit the projection μ and its additional good properties). For example, in [1, Example 3.3] the authors give a telling construction of a Clifford semigroup S with the following properties:

(1) each constituent group of S is amenable (in fact, can be taken to be the symmetric group on three objects);
(2) there exists a non-inner bounded derivation $\ell^1(S) \to \ell^1(S)$.

Thus in Theorem 7.2 the condition of commutativity is essential.

Normal bands.

Definition 7.3. A semigroup S is said to be a *band* if every element of S is an idempotent.

A commutative band is nothing but a semilattice, and one might hope that the techniques of [3] extend to show simplicial triviality for the ℓ^1-algebras of bands. While we have not been able to solve the general case, we can obtain positive results (in the sense of proving simplicial triviality) if we put further restrictions on our bands.

Definition 7.4. A band R is said to be rectangular if $abc = ac$ for all $a, b, c \in R$. More generally, a *normal band* is a band S which satisfies the identity

$$abca = acba \qquad (a, b, c \in S) .$$

Normal bands were introduced in [10], where it was observed that they are precisely the class of bands which arise as strong semilattices of rectangular bands. We may therefore apply the disintegration techniques developed in this paper.

Lemma 7.5. *Let R be a rectangular band. Then the chain complex $\mathcal{C}_*(\ell^1(R), \ell^1(R))$ is split by bounded linear maps whose norm is independent of R.*

Proof. This essentially follows from the '1-biprojectivity' of $\ell^1(R)$ – that is, the existence of a contractive $\ell^1(R)$-bimodule map $\rho : \ell^1(R) \to \ell^1(R) \widehat{\otimes} \ell^1(R)$ which is right inverse to the product map. We can in fact give an explicit sequence of splitting maps as follows: fix $z \in R$, and define $s_n : \mathcal{C}_n(\ell^1(R), \ell^1(R)) \to \mathcal{C}_{n+1}(\ell^1(R), \ell^1(R))$ to be the unique bounded linear map satisfying

$$s_n(e_{x_0} \otimes e_{x_1} \otimes \ldots \otimes e_{x_n}) := e_{x_0 z} \otimes e_{z x_0} \otimes e_{x_1} \otimes \ldots \otimes e_{x_n}$$

for every $x_0, x_1, \ldots, x_n$ in R. Clearly each s_n is contractive, and it is easily verified that $\mathsf{d}_n s_n + s_{n-1} \mathsf{d}_{n-1} = \mathsf{id}$ for all n. $\square$

Proposition 7.6. *Let S be a normal band. Then the convolution algebra $\ell^1(S)$ is simplicially trivial.*

Proof. Let $\mathcal{R}$ be the class of rectangular bands. By [7, Propn IV.5.14] S admits a strong semilattice decomposition of type $\mathcal{R}$, say $S \cong \coprod_{e \in L} R_e$.

By Theorem 3.6 the inclusion

$$(*) \qquad \overset{(1)}{\bigoplus}_{e \in L} \mathcal{C}_*(\ell^1(R_e), \ell^1(R_e)) \lhook\joinrel\longrightarrow \mathcal{C}_*(\ell^1(S), \ell^1(S))$$

induces isomorphism of homology groups; Lemma 7.5 implies that the complex on the left-hand side of $(*)$ is split exact in degrees 1 and above. $\qquad\square$

Remark. It seems likely that one could obtain a slightly more direct proof of this result. For, since one can construct 'free normal bands' on any given generating set, the construction of splitting maps for arbitrary normal bands should follow from the finite free case, using a recursive construction as in [3]. The key point is to check that one has a 'natural splitting map' in the sense of [3] in degree 0, in order to have a starting point for the induction.

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